A Tanaka formula for the derivative of intersection local time in R^1

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Abstract

Let B_t be a one dimensional Brownian motion, and let α' denote the derivative of the intersection local time of B_t as defined in [3]. The object of this paper is to prove the following formula

$$(0.1)\quad \frac{1}{2}\alpha'_t(x) + \frac{1}{2}sgn(x)t = \int_0^t L_s^{B_s - x} dB_s - \int_0^t sgn(B_t - B_u - x)du$$

which was given as a formal identity in [3] without proof.

Let B denote Brownian motion in R^1. In [3], Rosen demonstrated the existence of of a process which he termed the derivative of self intersection local time for B. That is, he showed that there is a process $\alpha_t(y)$, formally defined as

$$\alpha_t(y) = -\int_0^t \int_0^s \delta'(B_s - B_r - y)drds \tag{0.2}$$

such that, for any C^1 function g, we have

$$\int_0^t \int_0^s g'(B_s - B_r - y)drds = -\int_R g(y)\alpha'_t(y)dy \tag{0.3}$$

In this paper we'll prove a Tanaka-style formula for α' which was given without proof by Rosen in [3]. We define

$$sgn(x) = \begin{cases} -1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 1 & \text{if } x > 0 \end{cases} \tag{0.4}$$

Our result is

Theorem 1 *There is a set of measure one upon which the following holds for all x and t:*

$$\frac{1}{2}\alpha'_t(x) + \frac{1}{2}sgn(x)t = \int_0^t L_s^{B_s - x} dB_s - \int_0^t sgn(B_t - B_u - x)du \tag{0.5}$$

Proof: Fix t and x for the time being. In what follows, the constant c may change from line to line. Let $f(x) = \pi^{-1/2}e^{-x^2}$. Let $f_\varepsilon(x) = \frac{1}{\varepsilon}f(\frac{x}{\varepsilon})$, so that $f_\varepsilon \longrightarrow \delta$ weakly as $\varepsilon \longrightarrow 0$. We assume in all calculations below that $\varepsilon < 1$. Let

$$F_\varepsilon(x) = \int_0^x f_\varepsilon(t)dt \;\; = \;\; \int_0^{\frac{x}{\varepsilon}} f(t)dt \tag{0.6}$$

We apply Ito's formula to F_ε to get

$$\begin{aligned} F_\varepsilon(B_t - B_u - x) - F_\varepsilon(-x) = \int_u^t f_\varepsilon(B_s - B_u - x)dB_s + \\ \frac{1}{2}\int_u^t f'_\varepsilon(B_s - B_u - x)ds \end{aligned} \tag{0.7}$$

which gives

$$\begin{aligned} &\int_0^t F_\varepsilon(B_t - B_u - x)du - tF_\varepsilon(-x) \\ &= \int_0^t \int_0^s f_\varepsilon(B_s - B_u - x)dudB_s + \frac{1}{2}\int_0^t \int_u^t f'_\varepsilon(B_s - B_u - x)dsdu \end{aligned} \tag{0.8}$$

Note that $F_\varepsilon(x) \longrightarrow \frac{1}{2}sgn(x)$ as $\varepsilon \longrightarrow 0$. Furthermore, $|F_\varepsilon(x)| \leq \frac{1}{2}$ for all x, ε, so by the dominated convergence theorem, the first integral on the left approaches $\int_0^t sgn(B_t - B_u - x)du$ as $\varepsilon \longrightarrow 0$. By Theorem 1 in [3], the rightmost integral on the right side is equal to

$$-\frac{1}{2}\int_R f_\varepsilon(y - x)\alpha'_t(y)dy \tag{0.9}$$

This term approaches $-\frac{1}{2}\alpha'_t(x)$ as $\varepsilon \longrightarrow 0$ for all x at which $\alpha'_t(x)$ is continuous. In [3] it was shown that $\alpha'_t(x)$ is continuous for all $x \neq 0$. To deal with the case $x = 0$, we need another fact proved in [3], namely that $\alpha'_t(x) + sgn(x)$ is continuous in x. Using this, together with the fact that $f_\varepsilon(x)sgn(x)$ is an

odd function, we have the following string of equalities:

$$\begin{aligned}\lim_{\varepsilon\longrightarrow 0}\int_R f_\varepsilon(y)\alpha_t'(y)dy \\ = \lim_{\varepsilon\longrightarrow 0}\int_R f_\varepsilon(y)(\alpha_t'(y)+sgn(y))dy \\ = \alpha_t'(0)+sgn(0)=\alpha_t'(0)\end{aligned}\tag{0.10}$$

The only term which remains is the leftmost term on the right side of (0.8):

$$V(x,\varepsilon):=\int_0^t\int_0^s f_\varepsilon(B_s-B_u-x)dudB_s\tag{0.11}$$

We will show that

$$\int_0^s f_\varepsilon(B_s-B_u-x)du\longrightarrow L_s^{B_s-x}\tag{0.12}$$

in L^2, and this is enough to complete the proof for fixed x and t. Using the standard occupation times formula, we have a.s.

$$\int_0^s f_\varepsilon(B_s-B_u-x)du=\int f_\varepsilon(B_s-y-x)L_s^y dy\tag{0.13}$$

Since $\int f_\varepsilon=1$, we have

$$\begin{aligned}E[\int f_\varepsilon(B_s-y-x)L_s^y dy-L_s^{B_s-x}]^2 \\ \leq E[\int f_\varepsilon(B_s-y-x)|L_s^y-L_s^{B_s-x}|dy]^2 \\ \leq E[\int f_\varepsilon(B_s-y-x)|L_s^y-L_s^{B_s-x}|^2 dy]\end{aligned}\tag{0.14}$$

The last inequality is Jensen's inequality, as $f_\varepsilon(B_s-y-x)dy$ is a probability measure on R. We integrate separately over the two regions $\{|y-(B_s-x)|<\sqrt{\varepsilon}\}$ and $\{|y-(B_s-x)|\geq\sqrt{\varepsilon}\}$. We can bound the contribution to (0.14) from the second region by

$$2E[\int_{\{|y-(B_s-x)|\geq\sqrt{\varepsilon}\}} f_\varepsilon(B_s-y-x)(|L_s^y|^2+|L_s^{B_s-x}|^2)dy]\tag{0.15}$$

Expand this into the expectation of two integrals. The first is

$$E[\int_{\{|y-(B_s-x)|\geq\sqrt{\varepsilon}\}} f_\varepsilon(B_s-y-x)(L_s^y)^2 dy]\tag{0.16}$$

Since $|y-(B_s-x)| \geq \sqrt{\varepsilon}$, we see that $f_\varepsilon(B_s-y-x) \leq c(1/\varepsilon)e^{-1/\varepsilon}$. Thus, (0.16) is bounded by

$$c(1/\varepsilon)e^{-1/\varepsilon}\int E[(L_s^y)^2]dy \leq c(1/\varepsilon)e^{-1/\varepsilon} \tag{0.17}$$

We have used here the fact that $\int E[(L_s^y)^2]dy < \infty$. One way of proving this is to note that $E[(L_s^y)^2] \leq P(T_y < s)E[(L_s^0)^2]$ by the strong Markov property, where T_y is the first hitting time of y. $P(T_y < s) = P[|B_s| > |y|]$ by the reflection principle, and it is straightforward to check that

$$\int P[|B_s| > |y|]dy < \infty \tag{0.18}$$

Thus, (0.17) converges to 0 as $\varepsilon \longrightarrow 0$. The second integral is

$$\begin{aligned} E[|L_s^{B_s-x}|^2\int_{\{|y-(B_s-x)|\geq\sqrt{\varepsilon}\}} f_\varepsilon(B_s-y-x)dy] \\ \leq \int_{|y|>\sqrt{\varepsilon}} f_\varepsilon(y)dyE[|L_s^{B_s-x}|^2] \end{aligned} \tag{0.19}$$

We require the fact that $E[|L_s^{B_s-x}|^2]$ is finite, and this may be proved as follows:

$$\begin{aligned} L_s^{B_s-x} = \lim_{\varepsilon\longrightarrow 0}\int_0^s f_\varepsilon((B_s-B_u)-x)du = \\ \lim_{\varepsilon\longrightarrow 0}\int_0^s f_\varepsilon((B_s-B_{s-u})-x)du = \tilde{L}_s^x \end{aligned} \tag{0.20}$$

where $\tilde{L}_s^x$ is the local time of the Brownian motion $\tilde{B}_u = (B_s - B_{s-u})$. Then $E[(\tilde{L}_s^x)^2]$ is bounded by $E[(\tilde{L}_s^0)^2] < \infty$, for

$$\tilde{L}_s^x =_{law} 1_{\{\tilde{T}_x < s\}}\tilde{L}'^{\,0}_{s-\tilde{T}_x} \tag{0.21}$$

where $\tilde{T}_x$ is the first time $\tilde{B}_u$ hits x, and $\tilde{L}'$ is the local time of the Brownian motion $\tilde{B}'_u = \tilde{B}_{T_x+u} - \tilde{B}_{T_x}$. (0.21) is a.s. smaller than $\tilde{L}_s^0$, as local time is increasing in s. Thus, (0.19) is bounded by

$$c\int_{|x|>\sqrt{\varepsilon}} f_\varepsilon(x)dx = c\int_{|x|>\varepsilon^{-1/2}} f(x)dx \tag{0.22}$$

and this approaches 0 as $\varepsilon \longrightarrow 0$. We must now show that

$$E[\int_{|B_s-y-x|<\sqrt{\varepsilon}} f_\varepsilon(B_s-y-x)|L_s^y - L_s^{B_s-x}|^2dy] \tag{0.23}$$

approaches 0 as ε does. We will need the following lemma.

Lemma 1 *Given $\delta > 0$, there is an $M > 0$ such that*

$$E[(L_s^{B_s - x})^2 1_{\{|B_s - x| > M\}}] < \delta \tag{0.24}$$

Proof: By the Cauchy-Schwarz inequality, we have

$$E[(L_s^{B_s - x})^2 1_{\{|B_s - x| > M\}}] \leq E[(L_s^{B_s - x})^4]^{1/2} P(|B_s - x| > M)^{1/2} \tag{0.25}$$

Writing $\tilde{L}_s^x$ for $L_s^{B_s - x}$ as we have done before, we see

$$E[(L_s^{B_s - x})^4] = E[(\tilde{L}_s^x)^4] \leq E[(\tilde{L}_s^0)^4] \tag{0.26}$$

with the last inequality being due to the same argument as in steps (0.20) and (0.21). Local time at 0 has all moments, so $E[(L_s^{B_s - x})^4]$ is uniformly bounded. It is evident that $P(|B_s - x| > M) \longrightarrow 0$ as $M \longrightarrow \infty$. This proves the lemma. $\square$

Fix $\delta > 0$. The lemma, together with the fact that

$$E[(L_s^y)^2] \leq P(T_M < s) E[(L_s^0)^2] \tag{0.27}$$

when $y > M$, allows us to pick M sufficiently large so that

$$E[(L_s^y)^2], E[(L_s^{B_s - x})^2 1_{\{|B_s - x| > M\}}] < \delta \tag{0.28}$$

when $y > M$. Then, substituting $y' = y - (B_s - x)$

$$\begin{aligned}
&E[\int_{\{|B_s - y - x| < \sqrt{\varepsilon}\} \bigcap \{|y| > M+1\}} f_\varepsilon(B_s - y - x) |L_s^y - L_s^{B_s - x}|^2 dy] \\
&= E[\int_{\{|y'| < \sqrt{\varepsilon}\} \bigcap \{|y' + (B_s - x)| > M+1\}} f_\varepsilon(y') |L_s^{y' + (B_s - x)} - L_s^{B_s - x}|^2 dy'] \\
&\leq c \int_R f_\varepsilon(y') E[|L_s^{y' + (B_s - x)}|^2 1_{\{|y' + (B_s - x)| > M+1\}} + |L_s^{B_s - x}|^2 1_{\{|B_s - x| > M\}}] dy' \\
&\leq \delta c \int_R f_\varepsilon(y') dy' = \delta c
\end{aligned} \tag{0.29}$$

Therefore we can restrict the integral to the region $|y| < M+1$, which means $|B_s - x| < M + 2$. Now, by [1] there is an L^2 random variable $X(\omega)$ such that $|L_s^y - L_s^z| \leq X(\omega)|y - z|^k$, where $k > 0$ is any number less than $1/2$, whenever $|y|, |z| < M + 2$. Using this we have

$$\begin{aligned}
&E[\int_{\{|B_s - y - x| < \sqrt{\varepsilon}\} \bigcap \{|y| < M+1\}} f_\varepsilon(B_s - y - x) |L_s^y - L_s^{B_s - x}|^2 dy] \\
&\leq \varepsilon^{k/2} E[X(\omega)^2 \int_R f_\varepsilon(B_s - y - x) dy]
\end{aligned} \tag{0.30}$$

The dy integral is bounded by 1, so (0.30) is bounded by $\varepsilon^{2k}E[X(\omega)^2]$, and this converges to 0 as ε goes to 0. This proves that

$$(0.31) \qquad E[\int_{|B_s-y-x|<\sqrt{\varepsilon}} f_\varepsilon(B_s-y-x)|L_s^y - L_s^{B_s-x}|^2 dy] \longrightarrow 0$$

as $\varepsilon \longrightarrow 0$, and proves the result for fixed x, t. We would like to prove it to be true for all x, t on a set of full measure, however. We will do so by proving that, for $t, t' < M$ we have

$$(0.32) \qquad E[V(x,\varepsilon,t) - V(x',\varepsilon',t')]^{2n} \leq C_M|(x,\varepsilon,t)-(x',\varepsilon',t')|^{n/20}$$

for any positive integer $n \geq 3$. This will allow us to apply Kolmogorov's criteria(see [2], Theorem I.2.1) for uniform continuity, which will complete the proof. We will in fact show separately that

$$(0.33) \qquad E[V(x,\varepsilon,t) - V(x',\varepsilon,t)]^{2n} \leq C_M|x-x'|^{2n/3}$$

$$(0.34) \qquad E[V(x,\varepsilon,t) - V(x,\varepsilon',t)]^{2n} \leq C_M|\varepsilon-\varepsilon'|^{2n/3}$$

$$(0.35) \qquad E[V(x,\varepsilon,t) - V(x,\varepsilon,t')]^{2n} \leq C_M|t-t'|^{(n-1)/10}$$

and these clearly imply (0.32). In order to prove this, we'll need a convenient expression bounding $E(V(x,\varepsilon,t))^{2n}$. We'll use the identity

$$(0.36) \qquad f_\varepsilon(x) = \frac{i}{2\pi}\int_R e^{ixp}\hat{f}(\varepsilon p)dp$$

By the Burkholder-Davis-Gundy inequality(again see [2], Corollary IV.4.2) we have

$$\begin{aligned}
&E(V(x,\varepsilon,t))^{2n} \leq cE(\int_0^t (\int_0^s \int_R e^{i(B_s-B_u-x)p}\hat{f}(\varepsilon p)\,dp\,du)^2\,ds)^n \\
&= c\int_{R^{2n}}\int_{[0,t]^n}\int_{[o,s]^{2n}} (\prod_{i=1}^n \hat{f}(\varepsilon p_i)\hat{f}(\varepsilon p_i'))E[exp(i\sum_{i=1}^n [p_i(B_{s_i}-B_{u_i}) + p_i'(B_{s_i}-B_{u_i'})] \\
&exp(ix\sum_{i=1}^n p_i)exp(ix\sum_{i=1}^n p_i')(\prod_{i=1}^n du_i du_i' ds_i dp_i dp_i')
\end{aligned}$$

where i ranges from 1 to n in the products and sum. We will deal first with the variance in x and ε. We have the following bounds:

$$(0.37) \qquad |e^{ipx} - e^{ipx'}| \leq c|p|^{1/3}|x-x'|^{1/3}$$

$$(0.38) \qquad |\hat{f}(\varepsilon p) - \hat{f}(\varepsilon' p)| \leq c|p|^{1/3}|\varepsilon - \varepsilon'|^{1/3}$$

We will also use the trivial bounds $|e^{ipx}|, |\hat{f}(\varepsilon p)| \leq 1$. Thus,

$$E(V(x,\varepsilon,t) - V(x',\varepsilon,t))^{2n}$$
$$\leq c \int_{R^{2n}} \int_{[0,t]^n} \int_{[o,s]^{2n}} (\prod_{i=1}^n \hat{f}(\varepsilon p_i)\hat{f}(\varepsilon p_i'))E[exp(i\sum_{i=1}^n [p_i(B_{s_i} - B_{u_i}) + p_i'(B_{s_i} - B_{u_i'})]$$
$$(\prod_{i=1}^n |e^{ip_ix} - e^{ip_ix'}||e^{ip_i'x} - e^{ip_i'x'}|)(\prod_{i=1}^n du_i du_i' ds_i dp_i dp_i')$$
$$\leq c|x-x'|^{2n/3} \int_{R^{2n}} \int_{[0,t]^n} \int_{[o,s]^{2n}} E[exp(i\sum_{i=1}^n [p_i(B_{s_i} - B_{u_i}) + p_i'(B_{s_i} - B_{u_i'})]$$
$$(\prod_{i=1}^n |p_i|^{1/3}|p_i'|^{1/3})(\prod_{i=1}^n du_i du_i' ds_i dp_i dp_i')$$

Likewise,

$$E(V(x,\varepsilon,t) - V(x,\varepsilon',t))^{2n} \leq c \int_{R^{2n}} \int_{[0,t]^n} \int_{[o,s]^{2n}} (\prod_{i=1}^n exp(ix\sum_{i=1}^n p_i) exp(ix \sum_{i=1}^n p_i'))$$
$$E[exp(i\sum_{i=1}^n [p_i(B_{s_i} - B_{u_i}) + p_i'(B_{s_i} - B_{u_i'})](\prod_{i=1}^n |\hat{f}(\varepsilon p_i) - \hat{f}(\varepsilon' p_i)|)(\prod_{i=1}^n du_i du_i' ds_i dp_i dp_i')$$
$$\leq c|\varepsilon - \varepsilon'|^{2n/3} \int_{R^{2n}} \int_{[0,t]^n} \int_{[o,s]^{2n}} E[exp(i\sum_{i=1}^n [p_i(B_{s_i} - B_{u_i}) + p_i'(B_{s_i} - B_{u_i'})]$$
$$(\prod_{i=1}^n |p_i|^{1/3}|p_i'|^{1/3})(\prod_{i=1}^n du_i du_i' ds_i dp_i dp_i')$$

In order to control the variance in ε and x in the required (0.32) we need only bound

$$(0.39) \qquad c \int_{R^{2n}} \int_{[0,t]^n} \int_{[o,s]^{2n}} E[exp(i\sum_{i=1}^n [p_i(B_{s_i} - B_{u_i}) + p_i'(B_{s_i} - B_{u_i'})]$$
$$(\prod_{i=1}^n |p_i|^{1/3}|p_i'|^{1/3})(\prod_{i=1}^n du_i du_i' ds_i dp_i dp_i')$$

The value of the expectation in the integrand will depend on the ordering of the s_i's, u_i's, and u_i''s. For example, if $n = 2$, then while considering the

region $s_1 > s_2 > u_1 > u_1' > u_2 > u_2'$, we rewrite the integrand as

$$(0.40)\ E[exp(i[(p_1+p_1')(B_{s_1}-B_{s_2})+(p_1+p_1'+p_2+p_2')(B_{s_2}-B_{u_1})+ \\ (p_1'+p_2+p_2')(B_{u_1}-B_{u_1'})+(p_2+p_2')(B_{u_1'}-B_{u_2})+(p_2')(B_{u_2}-B_{u_2'})])]$$

By the independence of increments of Brownian motion, this expectation splits, and is equal to

$$\begin{aligned} &exp(-[(p_1+p_1')^2(s_1-s_2)+(p_1+p_1'+p_2+p_2')^2(s_2-u_1)+ \\ (0.41)\quad &(p_1'+p_2+p_2')^2(u_1-u_1')+(p_2+p_2')^2(u_1'-u_1)+(p_2')^2(u_1-u_2')]) \end{aligned}$$

We now substitute

$$(0.42)\ (z_1,z_2,z_3,z_4,z_5,z_6)=(s_1-s_2,s_2-u_1,u_1-u_1',u_1'-u_2,u_2-u_2',u_2')$$

and integrate with respect to the z_i's using the simple bound

$$(0.43)\qquad \int_0^t e^{-rb^2}dr \le \frac{c}{1+b^2}$$

We see that in order to show (0.39) is bounded in this case we must show

$$\begin{aligned} &\int_{R^4}\frac{(\prod_{i=1}^2|p_i|^{1/3}|p_i'|^{1/3})}{(1+(p_1+p_1')^2)(1+(p_1+p_1'+p_2+p_2')^2)(1+(p_1'+p_2+p_2')^2)} \\ (0.44)\quad &\frac{(\prod_{i=1}^2 dp_i dp_i')}{(1+(p_2+p_2')^2)(1+(p_2')^2)} \end{aligned}$$

is finite. Label the linear combinations of p_i's and p_i''s in the denominator as $v_1,...,v_5$. We see that

$$(0.45)\qquad (p_1,p_1',p_2,p_2')=(v_2-v_3,v_3-v_4,v_5-v_4,v_5)$$

Substituting these values into the integrand, we see that each v_j appears to a maximum power of 2/3 in the numerator. This implies that (0.39) is bounded by

$$(0.46)\qquad c\int_{R^4}(\prod_{j=1}^5\frac{1}{1+|v_j|^{4/3}})(\prod_{i=1}^2 dp_i dp_i')$$

We may transform linearly from (p_1,p_1',p_2,p_2') to (v_2,v_3,v_4,v_5), as is shown by (0.45).The resulting integral is finite, as the power of each variable in the denominator is greater than 1.

The general case may be handled in exactly the same way. That is, given an ordering of s_i's, p_i's, and p_i''s in (0.39), we may rewrite the expectation so that it factors as in (0.40). We substitute $(z_1, ..., z_{3n})$ for the differences of s_i's, p_i's and p_i''s, where z_{3n} is defined to be 0 to simplify what follows. We use the bound (0.43), and arrive at an expression of the form

$$\int_{R^{2n}} \frac{(\prod_{i=1}^n |p_i|^{1/3}|p_i'|^{1/3})(\prod_{i=1}^n dp_i dp_i')}{\prod_{j=1}^{3n}(1+|v_j|^2)} \tag{0.47}$$

Each p_i and p_i' can be expressed as $v_j - v_{j+1}$ for some $j \geq 1$. To see that this is true, note that when we rewrite the expectation, as in step (0.40), the only terms containing the u_i corresponding to a given p_i will be

$$... + v_j(B_a - B_{u_i}) + v_{j+1}(B_{u_i} - B_b) + ... \tag{0.48}$$

where a and b denote the s, u, or u' appearing immediately before or after u_i on the region to be integrated over. Comparing this with the coefficient of B_{u_i} in (0.39), we see that $p_i = v_j - v_{j+1}$. Let us denote by $j(i)$ and $j'(i)$ as the j values for which $p_i = v_{j(i)} - v_{j(i)+1}$ and $p_i' = v_{j'(i)} - v_{j'(i)+1}$. If we replace each p_i and p_i' in (0.47) by the correct v_j, we see that (0.47) is bounded by

$$\int_{R^{2n}} \frac{(\prod_{i=1}^n(|v_{j(i)}| + |v_{j(i)+1}|)^{1/3}(|v_{j'(i)}| + |v_{j'(i)+1}|)^{1/3})(\prod_{i=1}^n dp_i dp_i')}{\prod_{j=1}^{3n}(1+|v_j|^2)} \tag{0.49}$$

Each v_j appears at most twice in the numerator of (0.49), so (0.49) is bounded by

$$\int_{R^{2n}} \frac{(\prod_{i=1}^n dp_i dp_i')}{\prod_{j=1}^{3n}(1+|v_j|^{4/3})} \tag{0.50}$$

This is finite, as the set of v_j's spans the set of p_i's and p_i''s.

This handles the variance in x and ε. We must still control the variance in t. Assume $t' > t$. Then

$$\begin{aligned}
&E(V(x,\varepsilon,t) - V(x,\varepsilon,t'))^{2n} \\
&\leq c \int_{R^{2n}} \int_{[t,t']^n} \int_{[o,s]^{2n}} E[exp(i \sum_{i=1}^n [p_i(B_{s_i} - B_{u_i}) + p_i'(B_{s_i} - B_{u_i'})] \\
&(\prod_{i=1}^n du_i du_i' ds_i dp_i dp_i')
\end{aligned}$$

We will follow the steps (0.39) through (0.46). Note however that (0.43) may be combined with Hölder's inequality to obtain

$$\int_0^{t'-t} e^{-rb^2} dr \leq \frac{c|t-t'|^{1/p}}{(1+b^2)^{1/q}} \tag{0.51}$$

for any $p, q > 1$ such that $1/p + 1/q = 1$. We will use (0.51) in place of (0.43), with $q = 10/9, p = 10$. Since all of the s_i's must be restricted to the interval $[t, t']$, we will have at least $n-1$ of the z_k's restricted to $[0, t'-t]$(Recall that the z_k's are defined as in (0.42)). This shows that

$$E(V(x,\varepsilon,t) - V(x,\varepsilon,t'))^{2n}$$
$$\leq c|t-t'|^{(n-1)/10} \int_{R^{2n}} \frac{(\prod_{i=1}^n |p_i|^{1/3}|p_i'|^{1/3})(\prod_{i=1}^n dp_i dp_i')}{\prod_{i=1}^{3n}(1+|v_j|^2)^{9/10}}$$

Following the steps (0.49) and (0.52), the integral in (0.52) is bounded by

$$\int_{R^{2n}} \frac{(\prod_{i=1}^n dp_i dp_i')}{\prod_{j=1}^{3n}(1+|v_j|^{4/3-2/10})} \tag{0.52}$$

This integral is finite, so (0.33) is proved. We have therefore proved

$$E[V(x,\varepsilon,t) - V(x',\varepsilon',t')]^{2n} \leq C_M|(x,\varepsilon,t) - (x',\varepsilon',t')|^{n/20} \tag{0.53}$$

By Kolmogorov's continuity criterion, this implies that we may let $\varepsilon \longrightarrow 0$ to obtain a process which is defined on a set of full measure for all x, t. That process has already been proved to be almost surely equal to $\int_0^t L_s^{B_s - x} dB_s$ for each x, t. This completes the proof.

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